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Entropy of (2+1)-dimensional de Sitter black hole to all orders in the Planck length

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Abstract

We calculate the statistical entropy of a scalar field on the background of (2+1)-dimensional de Sitter space without an artificial cutoff considering corrections to all orders in the Planck length from a generalized uncertainty principle (GUP) on the quantum state density. The desired entropy proportional to the horizon perimeter is obtained.

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About three decades ago, Bekenstein had suggested that the entropy of a black hole is proportional to the area of horizon from a view point of information theory [1]. Subsequently, Hawking showed that black hole entropy satisfies exactly the area law by means of Hawking radiation based on quantum field theory [2]. After their works, 't Hooft investigated the statistical properties of a scalar field outside the horizon of Schwarzschild black hole by introducing an artificial brick wall cutoff [3] in order to remove the ultraviolet divergence near the horizon [4, 5, 6]. Recently, several authors [7, 8, 9] calculated the entropy of black holes to leading order in the Planck length by using a generalized uncertainty principle (GUP) [10, 11] solving the ultraviolet divergences of the just vicinity near horizon replacing a brick wall cutoff with a minimal length. In particular, by using a GUP we have studied the entropy of a scalar field on the (2+1)-dimensional de Sitter (DS) black hole background to leading order in the Planck length [12] improving the previous DS works having a brick wall [13]. However, Yoon et. al. have very recently pointed out that since a minimal length is actually related to a brick wall cutoff, the entropy integral about radial component r in the range near horizon should be carefully treated for a convergent entropy [14].

On the other hand, it is also well-known that the deformed Heisenberg algebra [10] leads to a GUP showing the existence of a minimal length [15, 16, 17, 18, 19, 20], which originates due to the quantum fluctuation of the gravitational field [21, 22]. Indeed, it has been shown that the Feynman propagator displays an exponential ultraviolet cutoff of the form of $\exp(-\lambda p^2)$, where the parameter $\sqrt{\lambda}$ actually plays a role of a minimal length as shown later. Moreover, quantum gravity phenomenology has been tackled with effective models based on GUPs and/or modified dispersion relations [23, 24] containing a minimal length as a natural ultraviolet cutoff [25]. The essence of ultraviolet finiteness of the Feynman propagator can be also captured by a nonlinear relation $p = f(k)$, where p and k are the momentum and the wave vector of a particle, respectively, generalizing the commutation relation between operators $\hat{x}$ and $\hat{p}$ to

$$[\hat{x}, \hat{p}] = i\hbar \frac{\partial p}{\partial k} \;\Leftrightarrow \Delta x \Delta p \geq \frac{\hbar}{2} \left| \;\left\langle \frac{\partial p}{\partial k} \right\rangle\; \right| \tag{1}$$

at quantum mechanical level [24]. Recently, Nouicer has investigated the GUP effects to all orders in the Planck length on black hole thermodynamics [15] by arguing that the GUP up to leading order correction in the Planck length is not enough because the wave vector k does not satisfy the asymptotic property in the modified dispersion relation [24].

Moreover, he has extended the calculation of entropy to all orders in the Planck length for (4+1)-dimensional Randall-Sundrum brane model [16]. Very recently, we have extended the calculation of entropy to all orders in the Planck length for (3+1)-dimensional Schwarzschild case by carefully considering the entropy integral about r in the range near horizon [17].

In this paper, we study the entropy to all order corrections in the Planck length of a scalar field on (2+1)-dimensional DS black hole background carefully considering the entropy integral about r in the range $(r_H - \epsilon, r_H)$ near horizon. Contrary to our general expectation, this study of a lower 3D case is not so trivial in contrast to the previous result of 4D Schwarzschild case [17]. By using the novel equation of states of density [16] motivated by the GUP in quantum gravity, we calculate the quantum entropy of a massive scalar field on (2+1)-dimensional DS black hole background introducing the incomplete Γ-functions and carrying out numerical calculations. As a result, we obtain the desired Bekenstein-Hawking entropy without any artificial cutoff and little mass approximation satisfying the asymptotic property of the wave vector k in the modified dispersion relation. From now on, we take the units as $G = \hbar = c = k_B \equiv 1$.

Let us start with the following action [12, 13]

$$I = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left[R - \frac{2}{l^2} \right], \tag{2}$$

where $\Lambda = 1/l^2$ is a cosmological constant. Then, the classical equation of motion yields the DS metric as

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 \tag{3}$$

with $f(r) = (1 - r^2/l^2)$. The horizon is located at $r = r_H = l$ and our spacetime is bounded by the horizon as the two-dimensional cavity within the inner space of the horizon $(0 \le r \le l)$. The inverse of Hawking temperature is given by $\beta_H = 2\pi l$.

In this DS background, let us consider a scalar field with mass μ, which satisfies the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) - \mu^2 \Phi = 0. \tag{4}$$

Substituting the wave function $\Phi(r, \theta, t) = e^{-i\omega t} \Psi(r, \theta)$, we find that this equation becomes

$$\frac{\partial^2 \Psi}{\partial r^2} + \left(\frac{1}{f} \frac{\partial f}{\partial r} + \frac{1}{r} \right) \frac{\partial \Psi}{\partial r} + \frac{1}{f} \left(\frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\omega^2}{f} - \mu^2 \right) \Psi = 0. \tag{5}$$

By using the Wenzel-Kramers-Brillouin approximation [3] with $\Psi \sim exp[iS(r, \theta)]$, we have

$$p_r{}^2 = \frac{1}{f} \left(\frac{\omega^2}{f} - \mu^2 - \frac{p_\theta^2}{r^2} \right), \tag{6}$$

where $p_r = \frac{\partial S}{\partial r}$ and $p_\theta = \frac{\partial S}{\partial \theta}$. On the other hand, we also obtain the square module momentum

$$p^2 = p_i p^i = g^{rr} {p_r}^2 + g^{\theta\theta} {p_\theta}^2 = \omega^2/f - \mu^2 \tag{7}$$

with the condition $\omega \geq \mu\sqrt{f}$. Moreover, considering the modified dispersion relation given by (1), the usual momentum measure $\prod_{i=1}^{n} dp^i$ is deformed to

$$\prod_{j=1}^{n} dp^j \prod_{i=1}^{n} \frac{\partial k^i}{\partial p^j}. \tag{8}$$

For simplicity, we will restrict ourselves to the isotropic case in one space-like dimension in the following. Then, according to the Refs. [19, 20], we have

$$\frac{\partial p}{\partial k} = e^{\lambda p^2}, \tag{9}$$

where λ is a dimensionless constant of order one in the Planck length units.

Now, let us consider the deformed algebra given by $[X, P] = i\ e^{\lambda P^2}$ with the representations $X \equiv i\ e^{\lambda P^2} \partial_p$ and $P \equiv p$ of the position and momentum operators, respectively. Then, this algebra leads to the generalized uncertainty relation including all order corrections in the Planck length as

$$\Delta X \Delta P \geq \frac{1}{2} \left\langle e^{\lambda P^2} \right\rangle \geq \frac{1}{2} e^{\lambda \left((\Delta P)^2 + \langle P \rangle^2 \right)}. \tag{10}$$

Note that $\langle P^{2n} \rangle \geq \langle P^2 \rangle^n$ and $(\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2$.

Next, in order to investigate the quantum implications of this deformed algebra, let us solve the above relation (10) for ΔP that is satisfied with the equality sign. Using the definition of the function of $W(\xi) \equiv -2\lambda\,(\Delta P)^2$ and the Lambert function of $W\,(\xi)\, e^{W(\xi)} = \xi$ [26], we obtain the momentum uncertainty as follows

$$\Delta P = \frac{e^{\lambda \langle P \rangle^2}}{2\Delta X} e^{\lambda (\Delta P)^2}. \tag{11}$$

In order to have a real physical solution for ΔP of the Lambert function, the argument of the Lambert function is required to satisfy $\xi \geq -1/e$, which naturally leads to the position uncertainty as

$$\Delta X \geq \sqrt{e\lambda/2}\ e^{\lambda \langle P \rangle^2} \equiv \Delta X_{\min}. \tag{12}$$

Here, $\Delta X_{\min}$ is a minimal uncertainty in position. Moreover, this minimal length intrinsically derived for physical states with $\langle P \rangle = 0$ is given by

$$\Delta X_0^A = \sqrt{e\lambda/2}, \tag{13}$$

which is the absolutely smallest uncertainty in position. In fact, this minimal length plays a role of a brick wall cutoff effectively giving the thickness of the thin-layer near the horizon [7, 8, 9, 12]. Furthermore, a series expansion of Eq. (11) with $\langle P \rangle = 0$ naturally includes the well-known form of the GUP up to the leading order correction in the Planck length units [9, 12] as follows

$$\Delta X \Delta P \approx \frac{1}{2}\left[1 + \lambda\,(\Delta P)^2 + \mathcal{O}\left((\Delta P)^4\right)\right]. \tag{14}$$

Then, the minimal length up to the leading order is given by $\Delta X_0^L = \sqrt{\lambda} \; < \; \Delta X_0^A$, where the superscripts "L" and "A" denote the leading order and all orders, respectively. However, only this leading order correction of the GUP does not satisfy the property that the wave vector k asymptotically reaches the cutoff in large energy region as recently reported in Ref. [24].

Now, let us calculate the statistical entropy of a scalar field on the (2+1) DS black hole background to all orders in the Planck length by using the GUP. When the gravity is turned on, the number of quantum states in a volume element in phase cell space based on the GUP in the (2+1)-dimensional DS space is given by

$$dn_A = \frac{d^2x d^2p}{(2\pi)^2} e^{-\lambda p^2}, \tag{15}$$

where $p^2 = p^i p_i$ $(i = r, \theta)$ and one quantum state corresponding to a cell of the volume is changed from $(2\pi)^2$ into $(2\pi)^2 e^{\lambda p^2}$ in the phase space [15, 16, 17].

From Eq. (15), the number of quantum states with the energy less than ω is given by

$$\begin{aligned} n_A(\omega) &= \frac{1}{(2\pi)^2}\int dr d\theta dp_r dp_\theta e^{-\lambda p^2} \\ &= \frac{1}{2}\int dr \frac{r}{\sqrt{f}}\left(\frac{\omega^2}{f} - \mu^2\right) e^{-\lambda(\frac{\omega^2}{f} - \mu^2)}. \end{aligned} \tag{16}$$

On the other hand, for the bosonic case the free energy at the inverse temperature β is given by

$$F_A = \frac{1}{\beta}\sum_K \ln\left[1 - e^{-\beta\omega_K}\right], \tag{17}$$

where K represents the set of quantum numbers. By using Eq. (16), the free energy can be rewritten as

$$F_A \;\approx\; \frac{1}{\beta}\int dn_A(\omega)\;\ln\left[1 - e^{-\beta\omega}\right]$$

$$= -\int_{\mu\sqrt{f}}^{\infty} d\omega \frac{n_A(\omega)}{e^{\beta\omega}-1}$$

$$= -\frac{1}{2}\int_{r_H-\epsilon}^{r_H} dr \frac{r}{f\sqrt{f}} \int_0^{\infty} d\omega \frac{\omega^2 e^{-\lambda\frac{\omega^2}{f}}}{(e^{\beta\omega}-1)}. \tag{18}$$

Here, we have taken the continuum limit in the first line and integrated it by parts in the second line. Furthermore, in the last line of Eq. (18), since $f \to 0$ near the event horizon, *i.e.*, in the range $(r_H - \epsilon, r_H)$, $\omega^2/f - \mu^2$ in $n_A(\omega)$ becomes ω^2/f although we do not require the little mass approximation.

Moreover, we are only interested in the contribution from the just vicinity near the horizon, $(r_H - \epsilon, r_H)$, which corresponds to a proper distance of order of the minimal length, $\sqrt{e\lambda/2}$. This is because the entropy closes to the upper bound only in this vicinity, which it is just the vicinity neglected by the brick wall method [3, 4, 5, 6]. Then, we have

$$\sqrt{\frac{e\lambda}{2}} = \int_{r_H-\epsilon}^{r_H} \frac{dr}{\sqrt{f(r)}} \approx \sqrt{\frac{2\epsilon}{\kappa}}, \tag{19}$$

where κ is the surface gravity at the horizon of the black hole and it is identified as $\kappa = \frac{1}{2}\frac{df}{dr}|_{\beta=\beta_H} = 2\pi{\beta_H}^{-1}$. Note that the Taylor's expansion of $f(r)$ near the horizon is given by $f(r) \approx 2\kappa(r_H - r) + \mathcal{O}\left((r_H - r)^2\right)$.

Before calculating the entropy, let us mention that Yoon et. al. have recently suggested that since the minimal length parameter λ is related to the brick wall cutoff ϵ in Eq. (19), the entropy integral about r in the range near the horizon should be carefully treated for a convergent entropy [14]. In particular, although the term $(e^{\beta\omega} - 1)$ in Eq. (18) with $x = \sqrt{\frac{\lambda}{f}}\omega$ was expanded in the previous works giving $\beta\sqrt{\frac{f}{\lambda}}x$, one may not simply expand up to the first order because the upper bound in the near horizon is independent of ϵ as shown in the relation $0 \le \frac{f}{\lambda} = \frac{2\kappa(r_H-r)}{\lambda} \le \frac{2\kappa\epsilon}{\lambda} = \kappa^2$ [17].

Now, let us carefully consider the integral about r near the horizon by extracting out the ϵ-factor through the Taylor's expansion of $f(r)$. Then, the free energy of F_A in Eq. (18) can be written as

$$F_A \approx -\frac{1}{2}\int_0^{\infty} d\omega \frac{\omega^2}{(e^{\beta\omega}-1)} \Lambda_A(\omega, \epsilon), \tag{20}$$

where $\Lambda_A(\omega, \epsilon)$ is defined by

$$\Lambda_A \equiv \int_{r_H-\epsilon}^{r_H} dr \frac{r}{(2\kappa(r_H - r))^{3/2}} e^{-\frac{\lambda\omega^2}{2\kappa(r_H-r)}}. \tag{21}$$

By defining $t = \frac{\lambda\omega^2}{2\kappa(r_H - r)}$, Λ_A becomes

$$\begin{aligned}
\Lambda_A &= \frac{1}{2\kappa}\int_\xi^\infty dt \left(\frac{r_H}{\sqrt{\lambda\omega^2}} t^{-1/2} - \frac{\sqrt{\lambda\omega^2}}{2\kappa} t^{-3/2}\right) e^{-t} \\
&= \frac{r_H}{2\kappa\sqrt{\lambda\omega^2}}\Gamma\left(\frac{1}{2},\ \xi\right) - \frac{\sqrt{\lambda\omega^2}}{(2\kappa)^2}\Gamma\left(-\frac{1}{2},\ \xi\right), \qquad (22)
\end{aligned}$$

where the incomplete Γ-functions are given as

$$\Gamma(a,\xi) = \int_\xi^\infty t^{a-1} e^{-t} dt, \quad \xi \equiv \frac{\lambda\omega^2}{2\kappa\epsilon}. \qquad (23)$$

Then, the entropy is given by

$$\begin{aligned}
S_A &= \beta^2 \frac{\partial F_A}{\partial \beta}\ |_{\beta=\beta_H} \\
&\approx \frac{1}{4}\left(\frac{\delta_1}{2\pi^2\sqrt{\lambda}}\right)(2\pi r_H) - \frac{\delta_2}{8\pi^3 r_H}\sqrt{\lambda}, \qquad (24)
\end{aligned}$$

where the numerical values δ_1 and δ_2 are given by

$$\begin{aligned}
\delta_1 &= \int_0^\infty dy \frac{y^2}{\sinh^2 y}\Gamma\left(\frac{1}{2},\ \frac{2y^2}{e\pi^2}\right) \approx 2.02, \\
\delta_2 &= \int_0^\infty dy \frac{y^4}{\sinh^2 y}\Gamma\left(-\frac{1}{2},\ \frac{2y^2}{e\pi^2}\right) \approx 5.51. \qquad (25)
\end{aligned}$$

This is the all order corrected finite entropy based on the GUP. Furthermore, if we assume $\lambda = \alpha {l_P}^2$, where l_P is the Planck length, and in the system of the Planck units $l_P = 1$, then the entropy can be rewritten by the desired perimeter law as $S_A = \frac{1}{4}(2\pi r_H)$ with $\alpha = \frac{\delta_1^2}{4\pi^4}$ neglecting the second term, which consists of the product of $\sqrt{\lambda}$ and r_H^{-1}, in the large black hole limit.

On the other hand, in order to compare the dominant leading term in Eq. (24) with that of the usual approximation approach [7, 8, 9, 16], let us calculate the entropy in the usual coarse-grained approximation. In terms of the variable $x = \omega\sqrt{\lambda/f}$ and the fact that $e^{\beta\omega} - 1 = e^{\beta\sqrt{\frac{f}{\lambda}}x} - 1 \approx \beta\sqrt{\frac{f}{\lambda}}x$ for $f \to 0$, we have

$$\begin{aligned}
F_A^o &= -\frac{1}{2\lambda\beta}\int_{r_H-\epsilon}^{r_H} dr \frac{r}{\sqrt{f}}\int_0^\infty dx\ x\ e^{-x^2} \\
&= -\frac{1}{4\lambda\beta}\int_{r_H-\epsilon}^{r_H} dr \frac{r}{\sqrt{f}}. \qquad (26)
\end{aligned}$$

Then, when $r \to r_H$, we get the entropy to all orders from the free energy (26) as follows

$$S_A^o = \beta^2 \frac{\partial F_A^o}{\partial \beta}|_{\beta=\beta_H}$$

$$
\begin{aligned}
&= \frac{1}{4\lambda}\int_{r_H-\epsilon}^{r_H} dr \frac{1}{\sqrt{f}}\, r \approx \frac{1}{4\lambda}\sqrt{\frac{e\lambda}{2}}\, r_H \\
&= \frac{1}{4}\left(\frac{\sqrt{e}}{2\sqrt{2}\pi}\frac{1}{\sqrt{\lambda}}\right)(2\pi r_H). \qquad (27)
\end{aligned}
$$

Note that if we assume $\lambda = \alpha {l_P}^2$, where l_P is the Planck length, and in the system of the Planck units $l_P = 1$, then the entropy can be also rewritten by the desired perimeter law as $S_A^o = \frac{1}{4}(2\pi r_H)$ with $\alpha = \frac{e}{8\pi^2}$.

Finally, it seems to be appropriate to comment on the dominant leading term of the entropy (24), which is obtained through the Taylor expansion of $f(r)$ near the horizon, comparing with the entropy (27), which is obtained through the usual approximation approach. Although their values are different, we have obtained the desired Bekenstein-Hawking entropy by properly adjusting the minimal length parameter λ for both approximation approaches. Therefore, we may propose that although by carefully considering all order corrected GUP in the Planck length the entropy can be finely obtained, it is enough to practically use the simple usual approximation for the desired entropy due to the existence of the adjustable parameter α.

In summary, by using the generalized uncertainty principle, we have investigated the entropy to all orders in the Planck length of the massive scalar field on the (2+1)-dimensional de Sitter black hole background carefully considering the entropy integral about r in the range $(r_H - \epsilon, r_H)$ near the horizon. As a result, we have finely obtained the desired Bekenstein-Hawking entropy without any artificial cutoff and any little mass approximation satisfying the asymptotic property of the wave vector k in the modified dispersion relation.

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